IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Variable separation approach for a differential-difference system: special Toda equation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2004 J. Phys. A: Math. Gen. 37 2401

(http://iopscience.iop.org/0305-4470/37/6/029)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.65
The article was downloaded on 02/06/2010 at 19:52

Please note that terms and conditions apply.

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2401–2411 PII: S0305-4470(04)63576-1

Variable separation approach for a differential-difference system: special Toda equation

Xian-Min Qian[1,2,3], **S Y Lou**[1,4] **and Xing-Biao Hu**[3]

[1] Department of Physics, Shanghai Jiao Tong University, Shanghai, 200030, People's Republic of China
[2] Department of Physics, Shaoxing College of Arts and Sciences, Shaoxing, 312000, People's Republic of China
[3] State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific Engineering Computing, Academy of Mathematics and System Sciences, Academia Sinica, PO Box 2719, Beijing 100080, People's Republic of China
[4] Department of Physics, Ningbo University, Ningbo 315211, People's Republic of China

Received 15 May 2003, in final form 21 November 2003
Published 28 January 2004
Online at stacks.iop.org/JPhysA/37/2401 (DOI: 10.1088/0305-4470/37/6/029)

Abstract

A bilinear variable separation approach is used to construct some special solutions for a differential-difference Toda equation. The semi-discrete form of the continuous formula which describes some types of special solutions for many (2 + 1)-dimensional continuous systems is found for a suitable quantity of the differential-difference Toda equation. Thus abundant semi-discrete localized coherent structures are constructed by appropriately selecting the arbitrary functions.

PACS numbers: 02.30.Ik, 05.45.Yv

1. Introduction

It is very difficult to find explicit exact solutions for both nonlinear partial differential equations (PDEs) and nonlinear differential-difference equations (DDEs). In linear mathematical physics, the Fourier transform method and the variable separation approach are the two most effective ways to find exact solutions of linear equations. In nonlinear mathematical physics, the so-called inverse scattering transformation (IST) serves as a 'nonlinear' Fourier transform for nonlinear integrable models. However, it is very difficult to extend the variable separation approach to nonlinear cases even for integrable systems.

Recently, several kinds of 'variable separation' approaches have been established, i.e. the classical method, the differential Stäckel matrix approach [1], the geometric method [2], the ansatz-based methods [3, 2], the functional variable separation approach [4], the derivative-dependent functional variable separation approach [5] and the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints) [6].

 © 2004 IOP Publishing Ltd Printed in the UK

In the study of the exact solutions for the nonlinear systems, Hirota's bilinear direct method [7] (and trilinear method [8]) is an excellent simple approach for finding solutions such as multiple soliton solutions, rational solutions and τ function solutions. Recently, the bilinear and trilinear (multi-linear in general) forms have been used to find a variable separation approach for a diversity of $(2+1)$-dimensional systems, including the Davey–Stewartson (DS) equation, the Nizhnik–Novikov–Veselov (NNV) equation, the modified NNV equation, the asymmetric NNV equation, the asymmetric DS equation, the dispersive long wave equation, the Broer–Kaup–Kupershmidt (BKK) system, the higher-order BKK system, the nonintegrable $(2+1)$-dimensional KdV equation, the $(2+1)$-dimensional Burgers equation, the long wave–short wave interaction model, the Maccari system, the general (N + M)-component AKNS system and the (2 + 1)-dimensional sine-Gordon model [9–15]. In [11], a quite universal formula

$$U \equiv \frac{2(a_1a_2 - a_0a_3)q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 pq)^2} \tag{1}$$

is derived to describe some special solutions for some suitable physical quantities of all the (2 + 1)-dimensional multi-linear variable separation approach (MLVSA) solvable models mentioned above. In equation (1), a_0, a_1, a_2 and a_3 are arbitrary constants, p is an arbitrary function of $\{x, t\}$ for all the known MLVSA solvable models, while q of (1) may be an arbitrary function of $\{y, t\}$ for some of the MLVSA solvable models, or an arbitrary solution of a special equation (say, the Riccati equation or diffusion equation) for of some other MLVSA solvable models. Because of some arbitrary characteristics, lower-dimensional functions (such as p), have been included in the formula (1) by selecting them appropriately, abundant localized structures such as multiple solitoffs, dromions, lumps, breathers, instantons, peakons, compactons, foldons, ghostons, ring solitons and chaotic and fractal patterns have been found [11].

In this paper, we are interested in an important question: can the MLVSA be extended to solve some nonlinear DDEs? In sections 2 and 3, the bilinear variable separation approach (BLVSA), a special case of the MLVSA, is applied to a special differential-difference Toda equation (SDDTE). A semi-discrete form of the formula (1) for a suitable quantity of the SDDTE is given in section 4, from which abundant semi-discrete localized excitations are constructed and depicted. The last section contains the conclusions and discussions.

2. SDDTE and its generalized bilinear form

In nonlinear discrete and semi-discrete physical systems, the most famous and important systems are the so-called Toda systems which are widely used in physics [16]. In this paper, we only consider a special differential-difference Toda equation (SDDTE)

$$\begin{aligned} Q(n)_{yt} &= \exp[Q(n+1) - Q(n)][Q(n+1) + Q(n)]_y \\ &\quad - \exp[Q(n) - Q(n-1)][Q(n) + Q(n-1)]_y \end{aligned} \tag{2}$$

where $Q(n) \equiv Q(n, y, t)$ is a function of the discrete variable n and the continuous variables $\{y, t\}$. The SDDTE (2) was first derived by Cao *et al* in a remarkable paper [17]. Some interesting integrable properties of the SDDTE (2) have been given in [17, 18]. Although it is unclear whether the SDDTE (2) has any direct physical application, this equation seems mathematically interesting. For example, let us consider the continuous analogue of equation (2). Setting

$$Q(n, y, t) = \epsilon u\left(n\epsilon + 2\epsilon t, y, -\tfrac{1}{3}t\epsilon^3\right) \qquad n\epsilon + 2\epsilon t = x \qquad -\tfrac{1}{3}t\epsilon^3 = T \tag{3}$$

we have $Q(n \pm 1, y, t) = \epsilon u(x \pm \epsilon, y, T)$ and

$$Q(n \pm 1, y, t) = \epsilon u(x, y, T) \pm \epsilon^2 \frac{\partial}{\partial x} u + \frac{\epsilon^3}{2} \frac{\partial^2}{\partial x^2} u \pm \frac{\epsilon^4}{6} \frac{\partial^3}{\partial x^3} u + \cdots. \tag{4}$$

Substituting these expressions into (2), we have

$$\begin{aligned}
-\frac{\epsilon^2}{3}[-6u_{xy} + \epsilon^2 u_{yT}] - \frac{\epsilon}{6}[12u_y + 6\epsilon u_{xy} + 3\epsilon^2 u_{xxy} + \epsilon^3 u_{xxxy}]& \\
\times \exp\left(\frac{\epsilon^2}{6}[6u_x + 3\epsilon u_{xx} + \epsilon^2 u_{xxx} + O(\epsilon^3)]\right)& \\
- \frac{\epsilon}{6}[-12u_y + 6\epsilon u_{xy} - 3\epsilon^2 u_{xxy} + \epsilon^3 u_{xxxy}]& \\
\times \exp\left(\frac{\epsilon^2}{6}[6u_x - 3\epsilon u_{xx} + \epsilon^2 u_{xxx} + O(\epsilon^3)]\right)& \\
= -\frac{1}{3}(u_{yT} + u_{xxxy} + 6u_{xy}u_x + 6u_{xx}u_y)\epsilon^4 + O(\epsilon^5) = 0.&
\end{aligned} \tag{5}$$

From (5), we know that (2) possesses a continuous analogue

$$u_{yT} + u_{xxxy} + 6u_{xy}u_x + 6u_{xx}u_y = 0 \tag{6}$$

which is equivalent to the Ito equation [19] by a simple transformation.

To get some exact solutions of the SDDTE (2) via BLVSA, one can use the following dependent variable transformation to bilinearize it:

$$Q(n) = \rho(n) + \ln\left(\frac{f(n+1)}{f(n)}\right). \tag{7}$$

In the transformation (7), $\rho(n) \equiv \rho(n, t)$, the seed solution of the SDDTE, has been selected as an arbitrary function of $\{n, t\}$ for convenience later.

Substitution of the dependent variable transformation (7) into the SDDTE yields

$$(T_+ - 1)\left(\frac{D_y D_t f(n) \cdot f(n) - 2\exp[\rho(n) - \rho(n-1)] D_y \exp(D_n) f(n) \cdot f(n)}{2f(n)^2}\right) = 0 \tag{8}$$

where T_+ is a shift operator, i.e., $T_+ F(n) = F(n+1)$; Hirota's bilinear differential operator $D_y^m D_t^k$ and the bilinear difference operator $\exp(D_n)$ are defined by

$$D_y^m D_t^k a \cdot b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(y, t) b(y', t')\bigg|_{y=y', t=t'}$$

$$\exp(\delta D_n) a(n) \cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n} - \frac{\partial}{\partial n'}\right)\right] a(n) b(n)\bigg|_{n=n'} \equiv a(n+\delta) b(n-\delta).$$

Multiplying (8) by the inverse operator of $T_+ - 1$ leads to a generalized bilinear SDDTE

$$D_y D_t f(n) \cdot f(n) - J(n, t) D_y \exp(D_n) f(n) \cdot f(n) + R(y, t) f(n)^2 = 0 \tag{9}$$

where

$$J(n) \equiv 2\exp[\rho(n) - \rho(n-1)] \tag{10}$$

is an arbitrary function of $\{n, t\}$ and $R(y, t)$, the kernel of the difference operator $T_+ - 1$, is an arbitrary function of $\{y, t\}$.

3. Variable separation solutions of the SDDTE

In order to find some exact solutions of (9), similar to the continuous cases [9–11], we look for the solutions of (9) in the form

$$f(n) = a_0 + a_1 p(n,t) + a_2 q(y,t) + a_3 p(n,t)q(y,t) \tag{11}$$

where a_0, a_1, a_2 and a_3 are arbitrary constants and the variable separated functions $q(y,t) \equiv q$ and $p(n,t) \equiv p(n)$ are only functions of $\{y,t\}$ and $\{n,t\}$, respectively. Equation (11) looks like Hirota's two-soliton form when q and $p(n)$ are exponential functions. Substituting the ansatz (11) into (9), we have

$$\begin{aligned}-2(a_2 + a_3 p(n))^2 q_y q_t &+ 2(a_3 a_0 - a_2 a_1) q_y p(n)_t + 2(a_2 + a_3 p(n))(a_0 + a_1 p(n) \\ &+ a_2 q + a_3 p(n) q) q_{yt} - J(n)(p(n+1) - p(n-1))(a_3 a_0 - a_2 a_1) q_y \\ &+ R(a_0 + a_2 q + a_1 p(n) + a_3 q p(n))^2 = 0.\end{aligned} \tag{12}$$

Because $p(n)$ and $J(n)$ are functions of $\{n,t\}$, and q and R are functions of $\{y,t\}$, (12) can be separated into the following two equations,

$$q_t = (a_0 + a_2 q)^2 c_1 + (a_1 + a_3 q)^2 c_2 + (a_0 + a_2 q)(a_1 + a_3 q) c_3 \tag{13}$$

$$p_t(n) = (a_0 a_3 - a_1 a - 2)(c_2 - c_3 p(n) + c_1 p(n)^2) + \tfrac{1}{2} J(n)(p(n+1) - p(n-1)) \tag{14}$$

when the arbitrary function R is selected as

$$R = -2\big(c_1 a_2^2 + c_2 a_3^2 + c_3 a_2 a_3\big) q_y \tag{15}$$

with $c_1 \equiv c_1(t)$, $c_2 \equiv c_2(t)$ and $c_3 \equiv c_3(t)$ being arbitrary functions of t.

In principle, as long as the arbitrary functions c_1, c_2, c_3 and $\rho(n)$ (and then $J(n)$) are fixed, we can obtain the corresponding special solutions of (13) and (14) and then the special solutions of the SDDTE (2). However, it is still very difficult to solve the nonlinear DDE (14) for fixed nonzero $J(n)$. Fortunately, just as in the continuous cases [9–15], the arbitrariness of the function $J(n)$ allows us to treat the problem alternatively. Consider the function $p(n)$ as an arbitrary function of the variables n and t, and the function $J(n)$ is defined by

$$J(n) = \frac{2}{p(n-1) - p(n+1)}[(a_0 a_3 - a_1 a - 2)(c_2 - c_3 p(n) + c_1 p(n)^2) - p_t(n)]. \tag{16}$$

It should be pointed out that the Riccati equation (13) is totally the same as that of the asymmetric NNV equation [11]. To find some special solutions of (13), one may select the arbitrary functions appropriately. Here we list two special selections.

(1) If we write c_1, c_2 and c_3 as

$$c_1 = \frac{a_3^2 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} - \frac{a_3(a_1 + a_3 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{(a_1 + a_3 A_2)^2 A_{3t}}{(a_1 a_2 - a_0 a_3)^2 A_1} \tag{17}$$

$$c_2 = \frac{a_2^2 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} - \frac{a_2(a_0 + a_2 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{(a_0 + a_2 A_2)^2 A_{3t}}{(a_1 a_2 - a_0 a_3)^2 A_1} \tag{18}$$

$$c_3 = \frac{(a_0 a_3 + a_1 a_2 + 2 a_2 a_3 A_2) A_{1t}}{(a_1 a_2 - a_0 a_3)^2 A_1} - \frac{2 a_2 a_3 A_{2t}}{(a_1 a_2 - a_0 a_3)^2} + 2\frac{(a_0 + a_2 A_2)(a_1 + a_3 A_2) A_{3t}}{(a_1 a_2 - a_0 a_3)^2 A_1} \tag{19}$$

with $A_1 \equiv A_1(t)$, $A_2 \equiv A_2(t)$ and $A_3 \equiv A_3(t)$ being arbitrary functions of t, then the general solution of (13) with (17)–(19) reads

$$q = \frac{A_1}{A_3 + F_1(y)} + A_2 \tag{20}$$

where $F_1 \equiv F_1(y)$ is an arbitrary function of y.

(2) If we select c_1, c_2 and c_3 as

$$c_1 = \frac{a_3^2 b_{0t}}{(a_1a_2 - a_0a_3)^2} - \frac{a_3(a_1 + a_3b_0)b_{1t}}{(a_1a_2 - a_0a_3)^2 b_1} - \frac{\left[(a_1 + a_3b_0)^2 - b_1^2a_3^2\right]b_{2t}}{(a_1a_2 - a_0a_3)^2 b_1} \tag{21}$$

$$c_2 = \frac{a_2^2 b_{0t}}{(a_1a_2 - a_0a_3)^2} - \frac{a_2(a_0 + a_2b_0)b_{1t}}{(a_1a_2 - a_0a_3)^2 b_1} - \frac{\left[(a_0 + a_2b_0)^2 - a_2^2b_1^2\right]b_{2t}}{(a_1a_2 - a_0a_3)^2 b_1} \tag{22}$$

$$\begin{aligned} c_3 = {} & \frac{(a_0a_3 + a_1a_2 + 2a_2a_3b_0)b_{1t}}{(a_1a_2 - a_0a_3)^2 b_1} - \frac{2a_2a_3b_{0t}}{(a_1a_2 - a_0a_3)^2} \\ & + 2\frac{\left[(a_0 + a_2b_0)(a_1 + a_3b_0) - a_2a_3b_1^2\right]b_{2t}}{(a_1a_2 - a_0a_3)^2 b_1} \end{aligned} \tag{23}$$

with $b_0 \equiv b_0(t)$, $b_1 \equiv b_1(t)$ and $b_2 \equiv b_2(t)$ being arbitrary functions of t, then the general solution of (13) with (21)–(23) reads

$$q = b_1 \tanh\left(b_2 + F_2(y)\right) + b_0 \tag{24}$$

with $F_2 \equiv F_2(y)$ being an arbitrary function of y.

4. Abundant coherent structures for the SDDTE

Substituting all the results obtained in the last section into (7) we arrive at many kinds of exact solutions for the field Q of the SDDTE. In continuous cases, for every MLVSA solvable system listed in [11], there exists a quantity whose special solutions can be expressed by the formula (1). Naturally an important question arises: *is there a suitable quantity for the SDDTE such that it can be described by a suitable semi-discrete form of the formula (1)?*

Fortunately, it is straightforward to prove that if we define a quantity of the SDDTE as

$$u \equiv -2Q_y(n) \tag{25}$$

then

$$u = U(n) \equiv \frac{2q_y(a_2a_1 - a_3a_0)(p(n+1) - p(n))}{(a_0 + a_2q + a_1p(n) + a_3qp(n))(a_0 + a_2q + a_1p(n+1) + a_3qp(n+1))}. \tag{26}$$

Obviously, the function $U(n)$ defined in (26) is just a suitable semi-discrete form of the continuous quantity U given by (1). The function $U(n)$ is semi-discrete since it is discrete in one direction and continuous in the other direction.

Now starting from the semi-discrete quantity u expressed by (26), we can obtain abundant semi-discrete localized excitations for the SDDTE by selecting the arbitrary functions appropriately.

Detailed studies show that the semi-discrete localized structures for u are very similar to the continuous ones which have been discussed in [11]. So here, we will not discuss all the possible localized excitations but only list some particular examples.

Example 1. Resonant semi-discrete dromions and solitoff solutions

If we restrict the functions $p(n)$ and q of (26) as

$$p(n) = \sum_{i=1}^{N} \exp(k_in + \omega_it + x_{0i}) \equiv \sum_{i=1}^{N} \exp(\xi_i) \tag{27}$$

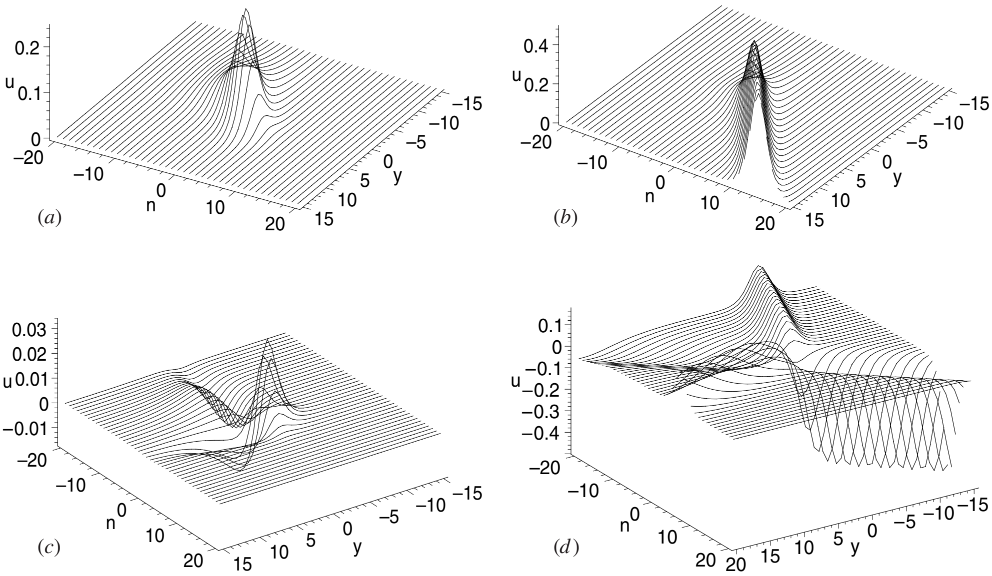

Figure 1. Four typical semi-discrete structures of SDDTE for the field u expressed by (25) with (27) and (28): (*a*) a special single-peak semi-discrete resonant dromion solution; (*b*) a single semi-discrete solitoff solution; (*c*) a multi-peak semi-discrete dromion solution and (*d*) a semi-discrete four-solitoff solution.

$$q = \sum_{i=1}^{M} \exp(K_i y + y_{0i}) \sum_{j=1}^{J} \exp(\Omega_j t + t_{0j}) \tag{28}$$

where x_{0i}, y_{0i}, t_{0j}, k_i, ω_i, K_i and Ω_i are arbitrary constants, M, N and J are arbitrary positive integers, then we have a single resonant semi-discrete dromion solution or semi-discrete multiple solitoff solutions. The selection (28) is related to the selections of the functions A_i $(i = 1, 2, 3)$, F_1 in (20) which are

$$A_3 = A_2 = 0 \tag{29}$$

$$A_1 = \sum_{j=1}^{J} \exp(\Omega_j t + t_{0j}) \tag{30}$$

$$F_1 = \frac{1}{\sum_{i=1}^{M} \exp(K_i y + y_{0i})} \tag{31}$$

and c_i $(i = 1, 2, 3)$ are given by (17)–(19) with (29) and (30).

In figure 1, we plot four typical structures caused by the resonant effects of four straight-line semi-discrete soliton solutions.

Figure 1(*a*) shows the structure of a first type of single resonant semi-discrete dromion solution expressed by (25) with (27), (28),

$$\begin{aligned} &M = N = 2 \qquad J = k_1 = K_1 = 1 \qquad k_2 = K_2 = \tfrac{1}{3} \\ &a_0 = 1 \qquad a_1 = a_2 = 10 \qquad a_3 = \tfrac{1}{2} \end{aligned} \tag{32}$$

and

$$x_{01} = y_{01} = t_{01} = x_{02} = y_{02} = 0 \tag{33}$$

at $t = 0$.

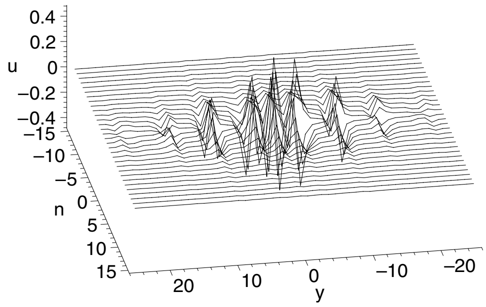

Figure 2. Plot of a special oscillating lump solution of the SDDTE for the quantity u expressed by (25) with (37) and (38) at $t = 0$.

Figure 1(*b*) is a plot of a single resonant semi-discrete solitoff solution shown by (25) with (27), (28), (33) and

$$\begin{aligned} &M = N = 2 \qquad J = k_1 = K_1 = 1 \qquad k_2 = K_2 = \tfrac{1}{3} \\ &a_0 = 1 \qquad a_1 = a_2 = 3 \qquad a_3 = 0 \qquad t = 0. \end{aligned} \tag{34}$$

Figure 1(*c*) reveals the structure of a second type of single semi-discrete resonant dromion solution described by (25) with (27), (28), (33) and

$$\begin{aligned} &M = N = 2 \qquad J = k_1 = -K_1 = 1 \qquad -k_2 = K_2 = \tfrac{1}{3} \\ &a_0 = a_2 = 10 \qquad a_3 = 1 \qquad a_1 = \tfrac{1}{2} \qquad t = 0. \end{aligned} \tag{35}$$

Figure 1(*d*) is a plot of a four-solitoff solution shown by (25) with (27), (28), (33) and

$$\begin{aligned} &M = N = 2 \qquad J = k_1 = -K_1 = 1 \qquad -k_2 = K_2 = \tfrac{1}{3} \\ &a_0 = 1 \qquad a_1 = a_2 = 3 \qquad a_3 = 0 \qquad t = 0. \end{aligned} \tag{36}$$

Example 2. Semi-discrete oscillating dromions and lumps

If some periodic functions in space variables are included in the functions $p(n)$ and q of (25), we may obtain some types of semi-discrete multi-dromion and multi-lump solutions with oscillating tails. The oscillating lump solution plotted in figure 2 is related to

$$q = \frac{1}{1 + [y(\cos(y) + 5/4)]^2} \qquad p(n) = \frac{1}{1 + (n - ct)^2} \tag{37}$$

$$a_0 = a_3 = 1 \qquad a_1 = a_2 = 5 \tag{38}$$

at $t = 0$.

Example 3. Multiple ring soliton solutions

In high dimensions, in addition to the point-like localized coherent excitations, there may be some other types of physically significant localized excitations. Recently, we have found some different kinds of ring soliton solutions which are not identically equal to zero at some closed two-dimensional and three-dimensional curves and decay exponentially away from the curves [15, 20–22].

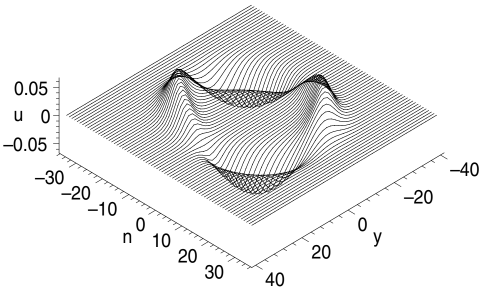

Figure 3. Plot of a typical single saddle type semi-discrete ring soliton solution for the quantity u of the SDDTE with the selections (39) and (40) at $t = 0$.

In figure 3, a typical saddle-type semi-discrete ring soliton solution is plotted for the quantity u with the selections

$$q = \exp\left(-\frac{y^2}{80} + 5\right) \qquad p(n) = \exp\left(\frac{(n - ct)^2}{80}\right) \tag{39}$$

and

$$a_0 = a_3 = 0 \qquad a_1 = a_2 = 5 \tag{40}$$

at $t = 0$.

Because of the existence of the arbitrariness in expression (25), there exist many kinds of multiple semi-discrete ring soliton solutions. Since the situations are quite similar to those in continuous cases, we do not discuss them further. However, it is worth mentioning that (as pointed out in [11, 12]) if the functions q and $p(n)$ are selected as the linear combinations of terms such as (39), then the interactions among the travelling ring soliton solutions are completely elastic without phase shifts. To find the multiple ring solitons with completely elastic interaction property and phase shifts, more complicated selections are needed [22].

5. Conclusions and discussions

In previous studies [9–15], we have successfully obtained rich classes of exact solutions for some famous (2 + 1)-dimensional nonlinear continuous integrable models via the MLVSA. In this paper, we have further applied the BLVSA to find some kinds of exact solutions of a special differential-difference Toda equation.

In continuous cases, a quite universal formula has been found to describe some types of special solutions of suitable fields or potentials of MLVSA solvable models. For the SDDTA, a semi-discrete form of the formula is obtained for a suitable quantity. An arbitrary function related to the discrete space variable is introduced in the semi-discrete form of the formula (1). Another function included in the formula is an arbitrary solution of the Riccati equation, and the Riccati equation is totally the same as that in some continuous MLVSA solvable models.

By selecting the arbitrary functions appropriately, one can construct abundant semi-discrete localized excitations such as multiple solitoffs, dromions, lumps, breathers, instantons and ring soliton solutions. The semi-discrete localized solutions are quite similar to those of continuous cases shown in [11].

In this paper, the special form of the MLVSA, BLVSA, is applied only to the SDDTE. Nonetheless, the method may be utilized for other higher-dimensional DDEs in a similar way. The basic steps are summarized as follows.

For a given DDE in the form of

$$F(x_i, n_j, u(x_i, n_j - k, \ldots), u_{x_i}(x_i, n_j - k, \ldots), \ldots, i = 1, \ldots, N_1, j = 1, \ldots, N_2,$$
$$k = 0, \pm 1, \pm 2, \ldots) \equiv F(u) = 0 \tag{41}$$

where $x_i, i = 1, 2, \ldots, N_1$, are continuous variables, $n_j, j = 1, 2, \ldots, N_2$, are discrete variables, the first step is to multi-linearize the original DDE (41) by using a suitable Bäcklund transformation. Usually, the resulting equations are bilinear equations for many integrable systems. In the second step, choose a seed solution of the Bäcklund transformation as general as possible in the sense that it contains one or more arbitrary functions. After that, a suitable variable separation ansatz with some variable separated functions has to be made. Usually, the ansatz is just the generalization of Hirota's two-soliton solution. Next, substitute the ansatz into the multi-linear equations and separate the resulting equations into several variable separated ones. The key step to get the variable separable solution is to solve the variable separated equations. Generally, to solve the variable separated PDEs and/or DDEs is still very difficult for any fixed seed solution. However, one can treat the problem in an inverse way: the variable separated functions appearing in the ansatz can be considered arbitrary and then the function(s) appearing in the seed solution will be fixed from the variable separated equations. Finally, from appropriate selections of the arbitrary functions included in the obtained variable separable solution, one may obtain abundant semi-discrete localized excitations of the DDE under investigation.

From the results of this paper for the SDDTE and the previous ones [9–15] for the PDEs, it is known that the BLVSA and MLVSA provide some reductions of the given nonlinear systems (say, (14) and (13) for SDDTE) and the reduction equations can be integrated quite trivially to provide a rich specific class of exact solutions. Usually, to reduce a high-dimensional PDE, one can use some kinds of symmetries. In the continuous case, we have known that the variable separable solutions obtained by the MLVSA can be recovered by the generalized conditional symmetry approach [26]. For the SDDTE, the situation is similar: finding the variable separable solution (7) with (11) and (10) via the generalized conditional symmetry approach is equivalent to finding a constraint equation

$$\eta(Q(n), Q(n \pm 1), \ldots, Q_y(n), \ldots) = 0 \tag{42}$$

such that (7) is not only a solution of the original nonlinear DDE (2) but also a solution of the generalized conditional symmetry constraint equation (42). Whence $Q(n)$ is a solution of both the SDDTE (2) and the constraint equation (42), we call

$$V \equiv \eta(Q(n), Q(n \pm 1), \ldots, Q_y(n), \ldots) \frac{\partial}{\partial Q(n)} \tag{43}$$

a generalized conditional symmetry of the SDDTE [3, 4]. For simplicity, we just write down the generalized conditional symmetry constraint equation for the bilinear SDDTE (9):

$$f_{yy}(n+1) f_y(n) - f_{yy}(n) f_y(n+1) = 0. \tag{44}$$

It is straightforward to find that f expressed by (11) is not only a solution of the original bilinear SDDTE but also a solution of (44). The corresponding generalized conditional symmetry constraint equation for the original field equation (2) can be obtained from (44) and (7). The result is more complicated and nonlocal.

Though the BLVSA (and the MLVSA) has been applied to many continuous integrable systems and the SDDTE, there still exist various important open problems which need to be studied further. For instance, since a continuous integrable system may possess some different types of integrable discrete forms, is the semi-discrete form of the formula (1) unique? In other words, how universal is the semi-discrete quantity (26)? In addition to the DDEs, there are various fully discrete systems in real physics. Can we extend the MLVSA to solve them? For some types of continuous systems, an arbitrary number of variable separated functions may be included by the Darboux transformations (i.e., for the NNV, asymmetric NNV and (2 + 1)-dimensional sine-Gordon models) [23] or by the extended MLVSA (i.e., for the BKK, the higher-order BKK, the dispersive long wave and the (2 + 1)-dimensional Burgers equations) [24]. How to extend the variable separable solution (26) for the SDDTE (perhaps also for other types of DDEs) such that an arbitrary number of variable separated functions can be included?

In recent studies of the nonlinear integrable systems, the τ function plays a crucial role. So it is interesting to construct the τ function solution related to the variable separable solution (26) for the SDDTE (2). However, we have to leave the problems for our future work because of the difficulties: (1) though the model does possess a Hirota-type bilinear representation, the coefficients of the bilinear equation are not constants. Instead, there are two arbitrary functions $J(n, t)$ and $R(y, t)$. (2) The existing problem of the whole SDDT hierarchy and then the general τ functions of the hierarchy have not yet been solved.

In the continuous case, though the MLVSA has been applied to various systems, some other important integrable systems, especially the Kadomtsev–Petviashvili equation and the Sawada–Kortera equation, have not yet been found suitable for the method. A similar situation occurs for DDEs. For instance, we still do not know whether the usual (2+1)-dimensional Toda system [25] can be solved by the MLVSA or not. More generally, it is still open how to exactly answer the general question: when can one apply the MLVSA successfully?

Acknowledgments

The authors are indebted to referees' comments and the helpful discussions with Drs Tang X Y, S-L Zhang and C-L Chen. The work is supported by the Outstanding Youth Foundation (no 19925522), the National Natural Science Foundation of China (no 90203001) and the Research Fund for the Doctoral Program of Higher Education of China (grant no 2000024832).

References

[1] Miller W 1977 *Symmetry and Separation of Variables* (Reading, MA: Addison-Wesley)
Kalnins E G and Miller W 1985 *J. Math. Phys.* **26** 1560
Kalnins E G and Miller W 1985 *J. Math. Phys.* **26** 2168
[2] Dolye P W and Vassiliou P J 1998 *Int. J. Non-linear Mech.* **33** 315
Dolye P W 1996 *J. Phys.A: Math. Gen.* **29** 7581
[3] Zhdanov R Z 1994 *J. Phys. A: Math. Gen.* **27** L291
Zhdanov R Z, Revenko I V and Fushchych W I 1995 *J. Math. Phys.* **36** 5506
Zhdanov R Z 1997 *J. Math. Phys.* **38** 1197
[4] Qu C Z, Zhang S L and Liu R C 2000 *Physica* D **144** 97
Estevez P G, Qu C Z, Zhang S L and Jhang 2002 *J. Math. Anal. Appl.* **275** 44
Zhang S L, Lou S Y and Qu C Z 2002 *Chin. Phys. Lett.* **12** 1741
[5] Zhang S L, Lou S Y and Qu C Z 2003 *J. Phys. A: Math. Phys.* **36** 12223
[6] Cao C W 1990 *Sci. China* A **33** 528
Cheng Y and Li Y S 1991 *Phys. Lett.* A **175** 22
Konopelchenko B G, Sidorenko V and Strampp W 1971 *Phys. Lett.* A **175** 17
Lou S Y and Chen L L 1999 *J. Math. Phys.* **40** 6491
[7] Hirota R 1971 *Phys. Rev. Lett.* **19** 1192
Sato M 1981 *RIMS Kokyuroku* **439** 30
Date E 1982 *J. Phys. Soc. Japan* **51** 4116
Jimbo M 1983 *Publ. Res. Math. Sci.* **19** 943

Kajiwara K 1991 *J. Math. Phys.* **32** 506
Tam H W and Hu X B 2003 *J. Phys. Soc. Japan* **72** 265
[8] Matsukidaira J, Satsuma J and Strampp W 1990 *Phys. Lett.* A **147** 467
Hietarinta J and Satsuma J 1991 *Phys. Lett.* A **161** 267
Satsuma J, Kajiwara K, Matsukidaira J and Hietarinta J 1992 *J. Phys. Soc. Japan* **61** 3096
[9] Lou S Y and Lu J Z 1996 *J. Phys. A: Math. Gen.* 29
Lou S Y 2000 *Phys. Lett.* A **277** 94
Lou S Y 2002 *Phys. Scr.* **65** 7
[10] Lou S Y and Ruan H Y 2001 *J. Phys. A: Math. Gen.* **34** 305
Tang X Y, Chen C L and Lou S Y 2002 *J. Phys. A: Math. Gen.* **35** L293
Lou S Y, Chen C L and Tang X Y 2002 *J. Math. Phys.* **43** 4078
Tang X Y 2003 *Phys. Lett.* A **314** 286
[11] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev.* E **66** 0466001
[12] Tang X Y and Lou S Y 2002 *Commun. Theor. Phys.* **38** 1
[13] Ruan H Y and Chen Y X 2001 *Acta Phys. Sin.* **4** 586 (in Chinese)
Huang W H and Zhang J F 2002 *Chin. Phys.* **11** 1101
Zhang J, Huang W and Zheng C 2002 *Acta Phys. Sin.* **51** 2676
[14] Lou S Y 2002 *J. Phys. A: Math. Gen.* **35** 10619
Lou S Y, Tang X Y, Qian X M, Chen C L, Lin J and Zhang S L 2002 *Mod. Phys. Lett.* B **16** 1075
[15] Lou S Y 2003 *J. Phys. A: Math. Gen.* **36** 3877
[16] Zinn-Justin P 2002 *Nucl. Phys.* B **634** 417
Coussement J, Kuijlaars A B J and Van Assche W 2002 *Inverse Probl.* **18** 923
Konopelchenko B G and Alonso L M 2002 *J. Math. Phys.* **43** 3807
Levi D and Winternitz P 2002 *J. Phys. A. Math. Gen.* **35** 2249
Svinin A K 2002 *J. Phys. A. Math. Gen.* **35** 2045
Viljoen H J, Lauderback L L and Sornette D 2002 *Phys. Rev.* E **65** 026609
[17] Cao C W, Geng X G and Wu Y T 1999 *J. Phys. A: Math. Gen.* **32** 8059
[18] Tam H-W, Hu X-B and Qian X-M 2002 *J. Math. Phys.* **43** 1008
[19] Ito M 1980 *J. Phys. Soc. Japan* **49** 286
[20] Lou S Y 2000 *J. Math. Phys.* **41** 6509
[21] Lou S Y, Yu J and Tang X Y 2000 *Z. Naturforsch.* **55** 867
[22] Lou S Y, Rogers C and Schief W K 2003 *J. Math. Phys.* **44** 5869
[23] Hu C, Lou S Y and Liu Q P 2003 *Chin. Phys. Lett.* **20** 1413
[24] Tang X Y and Lou S Y 2003 *Commun. Theor. Phys.* **40** 62
Tang X Y and Lou S Y 2003 *J. Math. Phys.* **44** 4000
[25] Olive D and Turok N 1986 *Nucl. Phys.* B **265** 469
Park Q H 1990 *Phys. Lett.* B **236** 423
Lou S Y 1993 *Phys. Rev. Lett.* **71** 4099
[26] Zhang S L, Tang X Y, Qian X M and Lou S Y in preparation